INVARIANCE OF PLURIGENERA OF VARIETIES WITH NONNEGATIVE KODAIRA DIMENSIONS

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Abstract

We prove the invariance of plurigenera under smooth projective deformations of projective varieties with nonnegative Kodaira dimensions. MSC32J25

Contents

1 Introduction

Let X be a smooth projective variety and let K_X be the canonical bundle of X. The canonical ring

$$R(X, K_X) := \oplus_{m \geq 0} H^0(X, \mathcal{O}_X(mK_X))$$

is a basic birational invariant of X. For every positive integer m, the m-th plurigenus $P_m(X)$ is defined by

$$P_m(X) := \dim H^0(X, \mathcal{O}_X(mK_X)).$$

It is believed that $P_m(X)$ is invariant under smooth projective deformation.

Recently Y.-T. Siu ([13]) proved that $P_m(X)$ is invariant under smooth projective deformation, if the all the fibers are of general type. This result has been slightly generalized by [6, 10].

The main idea of Siu is the comparison of the AZD on a special fiber and one on the total space of the deformation by using an induction on multiplier ideal sheaves.

In this paper, we consider the case of a smooth projective deformation such that all the fibers are smooth projective varieties with nonnegative Kodaira dimensions.

Theorem 1.1 *Let $\pi : X \longrightarrow \Delta$ be a smooth projective family of smooth projective varieties with nonnegative Kodaira dimensions over the open unit disk in the complex plane with center* 0. *Then for every positive integer m, the m-th plurigenus $P_m(X_t)(X_t := \pi^{-1}(t))$ is independent of $t \in \Delta$.*

We also prove the following theorem.

Theorem 1.2 *Let $\pi : X \longrightarrow \Delta$ be a smooth projective family of smooth projective varieties with pseudoeffective canonical bundles over the open unit disk Δ in the complex plane with center* 0. *Then the numerical Kodaira dimension $\kappa_{num}(X_t)$ is independent of $t \in \Delta$.*

Let us explain the idea of the proof of Theorem 1.1. Let $\pi : X \longrightarrow \Delta$ be the family as in Theorem 1.1. The heart of the proof is to construct a singular hermitian metric h on K_X such that Θ_h is semipositive and $h \mid_{X_t}$ is an AZD of K_{X_t} for every $t \in \Delta$. Then by the L^2-extension theorem ([11, p.200, Theorem]), we may easily deduce that $P_m(X_t)$ is independent of $t \in \Delta$.

Let A be an ample line bundle on X. Let m be an arbitrary positive integer. By the induction as in [13], it is easy to see that for a canonical AZD h_m of $A + mK_X$, $h_m \mid_{X_t}$ $(t \in \Delta)$ is an AZD of $\mathcal{O}_{X_t}(A + mK_X)$ for every $t \in \Delta$. But this is much weaker than what we want.

The key point of the proof is to embed X into a family of strongly pseudoconvex manifolds B as a divisor with the negative normal bundle $-A$. Then by the Bergman construction of an AZD of $K_B + X$ on B, we can construct the desired AZD of K_X.

This technique can be used to prove the abundance of canonical divisor of a projective manifold with nonnegative Kodaira dimension. This will be treated in the forthcoming paper.

2 Preliminaries

2.1 Multiplier ideal sheaves

In this subsection L will denote a holomorphic line bundle on a complex manifold M.

Definition 2.1 *A singular hermitian metric h on L is given by*

$$h = e^{-\varphi} \cdot h_0,$$

where h_0 is a C^∞-hermitian metric on L and $\varphi \in L^1_{loc}(M)$ is an arbitrary uppersemicontinuous function on M. We call φ a weight function of h.

The curvature current Θ_h of the singular hermitian line bundle (L, h) is defined by

$$\Theta_h := \Theta_{h_0} + \sqrt{-1}\partial\bar{\partial}\varphi,$$

where $\partial\bar{\partial}$ is taken in the sense of a current. The L^2-sheaf $\mathcal{L}^2(L, h)$ of the singular hermitian line bundle (L, h) is defined by

$$\mathcal{L}^2(L, h) := \{\sigma \in \Gamma(U, \mathcal{O}_M(L)) \mid h(\sigma, \sigma) \in L^1_{loc}(U)\},$$

where U runs over the open subsets of M. In this case there exists an ideal sheaf $\mathcal{I}(h)$ such that

$$\mathcal{L}^2(L, h) = \mathcal{O}_M(L) \otimes \mathcal{I}(h)$$

holds. We call $\mathcal{I}(h)$ the **multiplier ideal sheaf** of (L, h). If we write h as

$$h = e^{-\varphi} \cdot h_0,$$

where h_0 is a C^∞ hermitian metric on L and $\varphi \in L^1_{loc}(M)$ is the weight function, we see that

$$\mathcal{I}(h) = \mathcal{L}^2(\mathcal{O}_M, e^{-\varphi})$$

holds. For $\varphi \in L^1_{loc}(M)$ we define the multiplier ideal sheaf of φ by

$$\mathcal{I}(\varphi) := \mathcal{L}^2(\mathcal{O}_M, e^{-\varphi}).$$

Also we define

$$\mathcal{I}_\infty(h) := \mathcal{L}^\infty(\mathcal{O}_M, e^{-\varphi})$$

and call it the L^∞-**multiplier ideal sheaf** of (L, h).

Definition 2.2 *L is said to be pseudoeffective, if there exists a singular hermitian metric h on L such that the curvature current Θ_h is a closed positive current.*

Also a singular hermitian line bundle (L, h) is said to be pseudoeffective, if the curvature current Θ_h is a closed positive current.

It is easy to see that a line bundle L on a smooth projective manifold M is pseudoeffective, if and only if for an ample line bundle H on M, $L + \epsilon H$ is $\mathbf{Q}$-effective (or big) for every positive rational number ϵ (cf. [3]).

The following theorem is fundamental in the applications of multiplier ideal sheaves.

Theorem 2.1 *(Nadel's vanishing theorem [9, p.561]) Let (L, h) be a singular hermitian line bundle on a compact Kähler manifold M and let ω be a Kähler form on M. Suppose that Θ_h is strictly positive, i.e., there exists a positive constant ε such that*

$$\Theta_h \geq \varepsilon\omega$$

holds. Then $\mathcal{I}(h)$ is a coherent sheaf of $\mathcal{O}_M$ ideal and for every $q \geq 1$

$$H^q(M, \mathcal{O}_M(K_M + L) \otimes \mathcal{I}(h)) = 0$$

holds.

We note that the multiplier ideal sheaf of a singular hermitian $\mathbf{R}$-line bundle (i.e., a real power of a line bundle) is well defined because the multiplier ideal sheaf is defined in terms of the weight function. Sometimes it is useful to consider the following variant of multiplier ideal sheaves.

Definition 2.3 *Let h_L be a singular hermitian metric on a line bundle L. Suppose that the curvature of h_L is a positive current on X. We set*

$$\bar{\mathcal{I}}(h_L) := \lim_{\varepsilon \downarrow 0} \mathcal{I}(h_L^{1+\varepsilon})$$

and call it the closure of $\mathcal{I}(h_L)$.

As you see later, the closure of a multiplier ideal sheaf is easier to handle than the original multiplier ideal sheaf in some respect.

Next we shall consider the restriction of singular hermitian line bundles to subvarieties.

Definition 2.4 *Let h be a singular hermitian metric on L given by*

$$h = e^{-\varphi} \cdot h_0,$$

where h_0 is a C^{∞}-hermitian metric on L and $\varphi \in L^1_{loc}(M)$ is an uppersemi-continuous function. Here $L^1_{loc}(M)$ denotes the set of locally integrable functions (not the set of classes of almost everywhere equal locally integrable functions on M).

For a subvariety V of M, we say that the restriction $h \mid_V$ is well defined, if φ is not identically $-\infty$ on V.

Let (L, h),h_0,V, φ be as in Definition 2.4. Suppose that the curvature current Θ_h is bounded from below by some C^{∞}-(1,1)-form. Then φ is an almost plurisubharmonic function, i.e. locally a sum of a plurisubharmonic function and a C^{∞}-function. Let $\pi : \tilde{V} \longrightarrow V$ be an arbitrary resolution of V. Then $\pi^*(\varphi \mid_V)$ is locally integrable on $\tilde{V}$, since φ is almost plurisubharmonic. Hence

$$\pi^*(\Theta_h \mid_V) := \Theta_{\pi^* h_0 \mid_V} + \sqrt{-1}\partial\bar{\partial}\pi^*(\varphi \mid_V)$$

is well defined.

Definition 2.5 *Let φ be a plurisubharmonic function on a unit open polydisk Δ^n with center O. We define the Lelong number of φ at O by*

$$\nu(\varphi, O) := \liminf_{x \to O} \frac{\varphi(x)}{\log \mid x \mid},$$

where $\mid x \mid = (\sum \mid x_i \mid^2)^{1/2}$. Let T be a closed positive $(1,1)$-current on a unit open polydisk Δ^n. Then by $\partial\bar{\partial}$-Poincaré lemma there exists a plurisubharmonic function ϕ on Δ^n such that

$$T = \frac{\sqrt{-1}}{\pi}\partial\bar{\partial}\phi.$$

We define the Lelong number $\nu(T, O)$ at O by

$$\nu(T, O) := \nu(\phi, O).$$

It is easy to see that $\nu(T, O)$ is independent of the choice of ϕ and local coordinates around O. For an analytic subset V of a complex manifold X, we set

$$\nu(T, V) = \inf_{x \in V} \nu(T, x).$$

Remark 2.1 *More generally the Lelong number is defined for a closed positive (k, k)-current on a complex manifold.*

Theorem 2.2 *([12, p.53, Main Theorem]) Let T be a closed positive (k,k)-current on a complex manifold M. Then for every $c > 0$*

$$\{x \in M \mid \nu(T,x) \geq c\}$$

is a subvariety of codimension $\geq k$ in M.

The following lemma shows a rough relationship between the Lelong number of $\nu(\Theta_h, x)$ at $x \in X$ and the stalk of the multiplier ideal sheaf $\mathcal{I}(h)_x$ at x.

Lemma 2.1 *([1, p.284, Lemma 7][2],[12, p.85, Lemma 5.3]) Let φ be a plurisubharmonic function on the open unit polydisk Δ^n with center O. Suppose that $e^{-\varphi}$ is not locally integrable around O. Then we have that*

$$\nu(\varphi, O) \geq 2$$

holds. And if

$$\nu(\varphi, O) > 2n$$

holds, then $e^{-\varphi}$ is not locally integrable around O.

Let (L,h) be a pseudoeffective singular hermitian line bundle on a complex manifold M. The **closure** $\bar{\mathcal{I}}(h)$ of the multiplier ideal sheaf $\mathcal{I}(h)$ can be analysed in terms of Lelong numbers in the following way. We note that $\bar{\mathcal{I}}(h)$ is coherent ideal sheaf on M by Theorem 2.1.

In the case of $\dim M = 1$, we can compute $\bar{\mathcal{I}}(h)$ in terms of the Lelong number $\nu(\Theta_h, x)(x \in M)$. In fact in this case $\bar{\mathcal{I}}(h)$ is locally free and

$$\bar{\mathcal{I}}(h) = \mathcal{O}_M(-\sum_{x \in M}[\nu(\Theta_h, x)]x)$$

holds by Lemma 2.1, because $2 = 2\dim M$.

In the case of $\dim M \geq 2$, let $f : N \longrightarrow M$ be a modification such that $f^*\bar{\mathcal{I}}(h)$ is locally free. If we take f properly, we may assume that there exists a divisor $F = \sum_i F_i$ with normal crossings on Y such that

$$K_N = f^*K_M + \sum_i a_i F_i$$

and

$$\bar{\mathcal{I}}(h) = f_*\mathcal{O}_N(-\sum_i b_i F_i)$$

hold on Y for some nonnegative integers $\{a_i\}$ and $\{b_i\}$. Let $y \in F_i - \sum_{j \neq i} F_j$ and let $(U, z_1, \ldots, z_n)$ be a local corrdinate neighbourhood of y which is

biholomorphic to the open unit disk Δ^n with center O in $\mathbf{C}^n (n = \dim M)$ and

$$U \cap F_i = \{p \in U \mid z_1(p) = 0\}$$

holds. For $q \in \Delta^{n-1}$, we set $\Delta(q) := \{p \in U \mid (z_2(p), \ldots, z_n(p)) = q\}$. Then considering the family of the restriction $\{\Theta_h \mid_{\Delta(q)}\}$ for very general $q \in \Delta^{n-1}$, by Lemma 2.1, we see that

$$b_i = \max\{[\nu(f^*\Theta_h, F_i) - a_i], 0\}$$

holds for every i. In this way $\bar{\mathcal{I}}(h)$ is determined by the **Lelong numbers** of the curvature current on some modification. This is not the case, unless we take the closure as in the following example.

Example 1 *Let h_P be a singular hermitian metric on the trivial line bundle on the open unit polydisk Δ with center O in $\mathbf{C}$ defined by*

$$h_P = \frac{\| \cdot \|^2}{\mid z \mid^2 (\log \mid z \mid)^2}.$$

Then $\nu(\Theta_{h_P}, 0) = 1$ holds. But $\mathcal{I}(h_P) = \mathcal{O}_\Delta$ holds. On the other hand $\bar{\mathcal{I}}(h_P) = \mathcal{M}_0$ holds, where $\mathcal{M}_0$ is the ideal sheaf of $0 \in \Delta$.

2.2 Restriction of multiplier ideal sheaves to divisors

Let (L, h) be a pseudoeffective singular hermitian line bundle on a smooth projective variety X. Let D be a smooth divisor on X. We set

$$v_m(D) = \mathrm{mult}_D \mathrm{Spec}(\mathcal{O}_X / \mathcal{I}(h^m))$$

and

$$\tilde{\mathcal{I}}_D(h^m) = \mathcal{O}_D(v_m(D)D) \otimes \mathcal{I}(h^m).$$

Then $\tilde{\mathcal{I}}_D(h^m)$ is an ideal sheaf on D (it is torsion free, since D is smooth).

Let $x \in D$ be an arbitrary point of D and let $(U, z_1, \ldots, z_n)(n := \dim X)$ be a local coordinate neighbourhood of x which is biholomorphic to the unit open polydisk Δ^n with center O in $\mathbf{C}^n$ and

$$U \cap D = \{p \in U \mid z_1(p) = 0\}$$

holds. For $q \in \Delta^{n-1}$, we set $\Delta(q) := \{p \in U \mid (z_2(p), \ldots, z_n(p)) = q\}$. Then considering the family of the restriction $\{\Theta_h \mid_{\Delta(q)}\}$ for very general $q \in \Delta^{n-1}$, by Lemma 2.1, we see that

$$m \cdot \nu(\Theta_h, D) - 1 \leq v_m(D) \leq m \cdot \nu(\Theta_h, D)$$

holds.

We define the ideal sheaves $\sqrt[m]{\tilde{\mathcal{I}}_D(h^m)}$ on D by

$$\sqrt[m]{\tilde{\mathcal{I}}_D(h^m)_x} := \cup \mathcal{I}(\frac{1}{m}(\sigma))_x (x \in D),$$

where σ runs all the germs of $\tilde{\mathcal{I}}_D(h^m)_x$. And we set

$$\mathcal{I}_D(h) := \cap_{m \geq 1} \sqrt[m]{\tilde{\mathcal{I}}_D(h^m)}$$

and call it **the multipler ideal of** h **on** D. Also we set

$$\bar{\mathcal{I}}_D(h) := \lim_{\varepsilon \downarrow 0} \mathcal{I}_D(h^{1+\varepsilon}).$$

The following theorem is crucial in our proof of Theorem 1.1.

Theorem 2.3 *([17, Theorem 2.8]) Let (L, h) be a singular hermitian line bundle on a smooth projective variety X. Suppose that Θ_h is bounded from below by some negative multiple of a C^∞-Kähler form on X. Let D be a smooth divisor on X. If $h\mid_D$ is well defined, then*

$$\bar{\mathcal{I}}_D(h) = \bar{\mathcal{I}}(h\mid_D)$$

holds.

2.3 Analytic Zariski decompositions

In this subsection we shall introduce the notion of analytic Zariski decompositions. By using analytic Zariski decompositions, we can handle big line bundles like nef and big line bundles.

Definition 2.6 *Let M be a compact complex manifold and let L be a holomorphic line bundle on M. A singular hermitian metric h on L is said to be an analytic Zariski decomposition, if the followings hold.*

1. *Θ_h is a closed positive current,*
2. *for every $m \geq 0$, the natural inclusion*

   $$H^0(M, \mathcal{O}_M(mL) \otimes \mathcal{I}(h^m)) \to H^0(M, \mathcal{O}_M(mL))$$

   is an isomorphim.

Remark 2.2 *If an AZD exists on a line bundle L on a smooth projective variety M, L is pseudoeffective by the condition 1 above.*

Theorem 2.4 *([14, 15]) Let L be a big line bundle on a smooth projective variety M. Then L has an AZD.*

As for the existence for general pseudoeffective line bundles, now we have the following theorem.

Theorem 2.5 *([4]) Let X be a smooth projective variety and let L be a pseudoeffective line bundle on X. Then L has an AZD.*

Proof of Theorem 2.5. Let h_0 be a fixed C^∞-hermitian metric on L. Let E be the set of singular hermitian metric on L defined by

$$E = \{h; h : \text{lowersemicontinuous singular hermitian metric on } L,$$
$$\Theta_h \text{ is positive}, \frac{h}{h_0} \geq 1\}.$$

Since L is pseudoeffective, E is nonempty. We set

$$h_L = h_0 \cdot \inf_{h \in E} \frac{h}{h_0},$$

where the infimum is taken pointwise. The supremum of a family of plurisubharmonic functions uniformly bounded from above is known to be again plurisubharmonic, if we modify the supremum on a set of measure 0(i.e., if we take the uppersemicontinuous envelope) by the following theorem of P. Lelong.

Theorem 2.6 *([8, p.26, Theorem 5]) Let $\{\varphi_t\}_{t\in T}$ be a family of plurisubharmonic functions on a domain Ω which is uniformly bounded from above on every compact subset of Ω. Then $\psi = \sup_{t\in T} \varphi_t$ has a minimum uppersemicontinuous majorant ψ^* which is plurisubharmonic.*

Remark 2.3 *In the above theorem the equality $\psi = \psi^*$ holds outside of a set of measure* 0*(cf.[8, p.29]).*

By Theorem 2.6 we see that h_L is also a singular hermitian metric on L with $\Theta_h \geq 0$. Suppose that there exists a nontrivial section $\sigma \in \Gamma(X, \mathcal{O}_X(mL))$ for some m (otherwise the second condition in Definition 3.1 is empty). We note that

$$\frac{1}{\mid \sigma \mid^{\frac{2}{m}}}$$

gives the weihgt of a singular hermitian metric on L with curvature $2\pi m^{-1}(\sigma)$, where (σ) is the current of integration along the zero set of σ. By the construction we see that there exists a positive constant c such that

$$\frac{h_0}{\mid \sigma \mid^{\frac{2}{m}}} \geq c \cdot h_L$$

holds. Hence

$$\sigma \in H^0(X, \mathcal{O}_X(mL) \otimes \mathcal{I}_\infty(h_L^m))$$

holds. Hence in praticular

$$\sigma \in H^0(X, \mathcal{O}_X(mL) \otimes \mathcal{I}(h_L^m))$$

holds. This means that h_L is an AZD of L. **Q.E.D.**

Remark 2.4 *By the above proof we have that for the AZD h_L constructed as above*

$$H^0(X, \mathcal{O}_X(mL) \otimes \mathcal{I}_\infty(h_L^m)) \simeq H^0(X, \mathcal{O}_X(mL))$$

holds for every m.

The following proposition implies that the multiplier ideal sheaves of $h_L^m(m \geq 1)$ constructed in the proof of Theorem 2.5 are independent of the choice of the C^∞-hermitian metric h_0. The proof is trivial. Hence we omit it.

Proposition 2.1 *h_0, h_0' be two C^∞-hermitian metrics on a pseudoeffective line bundle L on a smooth projective variety X. Let h_L, h_L' be the AZD's constructed as in the proof of Theorem 2.5 associated with h_0, h_0' respectively. Then*

$$(\min_{x \in X} \frac{h_0}{h_0'}(x)) \cdot h_L' \leq h_L \leq (\max_{x \in X} \frac{h_0}{h_0'}(x)) \cdot h_L'$$

hold. In particular

$$\mathcal{I}(h_L^m) = \mathcal{I}((h_L')^m)$$

holds for every $m \geq 1$.

We call the AZD constructed as in the proof of Theorem 2.5 **a canonical AZD** of L. Proposition 2.1 implies that the multiplier ideal sheaves associated with the multiples of the canonical AZD are independent of the choice of the canonical AZD.

2.4 Numerical Kodaira dimension

The numerical Kodaira dimension was first introduced by Y. Kawamata in [6], for nef line bundles on a projective algebraic varieties. Here we give another definition of the numerical Kodaira dimension for any line bundles on projective algebraic varieties. This definition coincides with the former definition when the line bundle is nef.

Definition 2.7 *Let L be a line bundle on a smooth projective variety X. Let A be an ample line bundle on X. We define the numerical Kodaira dimension $\kappa_{num}(L)$ of L by*

$$\kappa_{num}(X,L) := \sup_{\ell \geq 1}(\limsup_{m\to\infty}\frac{\log \dim H^0(X,\mathcal{O}_X(\ell A + mL))}{\log m}).$$

And we define the numerical Kodaira dimension $\kappa_{num}(X)$ of X by

$$\kappa_{num}(X) := \kappa(X, K_X).$$

Remark 2.5 *It is clear that $\kappa_{num}(X,L) \geq 0$, if and only if L is pseudoeffective. If L is nef, then*

$$\kappa_{num}(X,L) = \sup\{k \mid c_1^k(L) \text{ is not numerically trivial}\}$$

holds. The righthandside is the definition of the numerical Kodaira dimension defined in [6].

It is clear that

$$\kappa_{num}(X,L) \geq \kappa(X,L)$$

holds.

3 Proof of Theorem 1.2

Let $\pi : X \longrightarrow \Delta$ be the family as in Theorem 1.2. Let A be an ample line bundle on X Let $h_{m,0}$ be a canonical AZD of $mK_{X_0} + A$ and let h_m be a canonical AZD of $mK_X + A$. Then we have the following lemma.

Lemma 3.1 *For every positive integer ℓ*

$$\bar{\mathcal{I}}(h_{m,0}^{\ell}) = \bar{\mathcal{I}}(h_m^{\ell} \mid_{X_0})$$

holds.

Proof of Lemma 3.1. We prove this lemma by induction on m. If $m = 0$, then the assertion is clear. Suppose that the assertion is settled for $m-1(m \geq 1)$. Let h_{m-1} be a canonical AZD of $(m-1)K_X + A$. Let ξ_{m-1} be the function such that

$$h_{m-1} = e^{-\xi_{m-1}} \cdot H_{m-1}$$

holds, where H_{m-1} is a C^∞-hermitian metric on $(m-1)K_X + A$. Let us fix a C^∞-hemitian metric H_m on $mL + A$ Let φ_0,φ be functions on X_0 defined by

$$h_{m,0} = e^{-\varphi_0} \cdot H_m$$

and

$$h_m \mid_{X_0} = e^{-\varphi} \cdot H_m.$$

Let L_m denotes the line bundle $mK_X + A$. Let E be an effective $\mathbf{Q}$-divisor on X such that $L_m - E$ is ample. Let ψ be a function on X such that

$$e^{-\psi} \cdot H_m$$

is a C^∞ hermitian metric on $L_m - E$.

There exists a positive integer a such that

$$M = a(L_m - E)$$

is Cartier and for any pseudoeffective singular hemitian line bundle (L, h_L) on X, $\mathcal{O}_X(M + L) \otimes \mathcal{I}(h_L)$ is globally generated. The existence of such a follows from [13, p.664, Proposition 1].

From now on we consider all the functions on X_0. The following lemma is clear.

Lemma 3.2 *We may assume that*

$$\psi \leq \varphi \leq \varphi_0 \leq \xi_{m-1}$$

hold, if we adjust them by adding real constants.

The following lemma is similar to [13, p.670, Proposition 5].

Lemma 3.3 *Let* $0 < \varepsilon << 1$ *be a sufficiently small positive number such that* $e^{-\varepsilon\psi}$ *is locally integrable every where on* X *and* $e^{-\varepsilon\psi} \mid_{X_0}$ *is locally integrable everywhere on* X_0*. Then*

$$\mathcal{I}((\ell - \varepsilon)\varphi_0 + (a + \varepsilon)\psi) \subset \mathcal{I}((\ell - 1 + a - \varepsilon)\varphi + \varepsilon\psi + \xi_{m-1})$$

holds for every positive integer ℓ.

Proof. The proof here is essentially same as in [13, p.670, Proposition 5]. We prove the lemma by induction on ℓ.

If $\ell = 1$,

$$\begin{aligned}(1-\varepsilon)\varphi_0 + (a+\varepsilon)\psi &\leq (1-\varepsilon)\varphi_0 + a\varphi + \varepsilon\psi \\ &\leq (a-\varepsilon)\varphi + \varepsilon\psi + \xi_{m-1}\end{aligned}$$

hold.

Suppose that the induction step ℓ has been settled. Let σ be an element of

$$H^0(X_0, \mathcal{O}_{X_0}((\ell+a+1)L_m) \otimes \mathcal{I}((\ell+1-\varepsilon)\varphi_0 + (a+\varepsilon)\psi).$$

Then by the induction assumption, we have that

$$\begin{aligned}&H^0(X_0, \mathcal{O}_{X_0}((\ell+a+1)L_m) \otimes \mathcal{I}((\ell+1-\varepsilon)\varphi_0 + (a+\varepsilon)\psi)) \\ &\subset H^0(X_0, \mathcal{O}_{X_0}((\ell+a+1)L_m) \otimes \mathcal{I}((\ell-\varepsilon)\varphi_0 + (a+\varepsilon)\psi)) \\ &\subset H^0(X_0, \mathcal{O}_{X_0}((\ell+a+1)L_m) \otimes \mathcal{I}((\ell-1+a-\varepsilon)\varphi + \varepsilon\psi + \xi_{m-1})) \\ &= H^0(X_0, \mathcal{O}_{X_0}(L_m + (\ell+a)L_m) \otimes \mathcal{I}((\ell-1+a-\varepsilon)\varphi + \varepsilon\psi + \xi_{m-1}))\end{aligned}$$

hold. We note that the singular hermitian metric

$$h_{m-1} = e^{-\xi_{m-1}} \cdot H_{m-1}$$

on $(m-1)K_X + A$ has semipositive curvature in the sense of a current. Then by the definitions of L_m and ξ_{m-1}, Nadel's vanishing theorem (Theorem 2.1) implies that

$$H^1(X, \mathcal{O}_X(L_m - X_0 + (\ell+a)L_m) \otimes \mathcal{I}((\ell-1+a-\varepsilon)\varphi + \varepsilon\psi + \xi_{m-1})) = 0$$

holds. Hence σ is the restriction of an element of

$$H^0(X, \mathcal{O}_X(L_m + (\ell+a)L_m) \otimes \mathcal{I}((\ell-1+a-\varepsilon)\varphi + \varepsilon\psi + \xi_{m-1})).$$

This implies that

$$\sigma \in H^0(X_0, \mathcal{O}_{X_0}((\ell+1+a)L_m) \otimes \mathcal{I}((\ell+a-\varepsilon)\varphi + \varepsilon\psi + \xi_{m-1}))$$

holds by the definition of φ, i.e., by the definition of an AZD. Since

$$H^0(X_0, \mathcal{O}_{X_0}((\ell+a+1)L_m) \otimes \mathcal{I}((\ell+1-\varepsilon)\varphi_0 + (a+\varepsilon)\psi)).$$

is globally generated by the construction of A, we conclude that

$$\mathcal{I}((\ell+1-\varepsilon)\varphi_0 + (a+\varepsilon)\psi) \subset \mathcal{I}((\ell+a-\varepsilon)\varphi + \varepsilon\psi + \xi_{m-1})$$

holds. This completes the proof of Lemma 3.3. **Q.E.D.**

Let

$$f : Y_0 \to X_0$$

be any modification. By Lemma 3.3 and Lemma 2.1, we see that

$$\nu(f^*((\ell - \varepsilon)\varphi_0 + (a + \varepsilon)\psi)) + n \geq \nu(f^*((\ell - 1 + a - \varepsilon)\varphi + \varepsilon\psi + \xi_{m-1}))$$

holds, where $n = \dim X_0$. Dividing the both sides by ℓ and letting ℓ tend to infinity, we have that

$$\nu(f^*\varphi_0) \geq \nu(f^*\varphi)$$

holds. On the other hand, we note that

$$\nu(f^*\varphi_0) \leq \nu(f^*\varphi)$$

holds by Lemma 3.2. Hence we see that for every $k \geq 0$,

$$\bar{\mathcal{I}}(k\varphi_0) = \bar{\mathcal{I}}(k\varphi)$$

holds on X_0, since the closure of a multiplier ideal sheaf is determined by the Lelong numbers of pullback of the curvature currents on some modification of the space as in Section 2.1. By the definitions of φ and φ_0, this completes the proof of Lemma 3.1. **Q.E.D.**

Now we shall prove Theorem 1.2. By Lemma 3.1 and the L^2-extension theorem ([11, p.200, Theorem]), we see that

$$H^0(X, \mathcal{O}_X(mK_X + 2A) \otimes \bar{\mathcal{I}}(h_m)) \to H^0(X_0, \mathcal{O}_{X_0}(mK_X + 2A) \otimes \bar{\mathcal{I}}(h_{m,0}))$$

is surjective for every $m \geq 0$. Since A can be replaced by its any positive multiple, we see that

$$\kappa_{num}(X_t) \leq \kappa_{num}(X_t)$$

for every very general $t \in \Delta$. On the other hand by the uppersemicontinuity of the cohomology groups, we see that the opposite inequality holds. Hence we see that $\kappa_{num}(X_t)$ is independent of $t \in \Delta$. This completes the proof of Theorem 1.2.

4 Proof of Theorem 1.1

Let $\pi : X \longrightarrow \Delta$ be the family as in Theorem 1.1. Let h be a canonical AZD of K_X and let h_0 be a canonical AZD of K_{X_0}.

4.1 Logarithmic reduction

Let A be an ample line bundle on X and let h_A be a C^∞-hermitian metric on X with strictly positive curvature. We may and do assume that for every pseudoeffective singular hermitian line bundle (F, h_F), $\mathcal{O}_X(F + A) \otimes \mathcal{I}(h_F)$ is globally generated (cf. [13, Proposition 1.1]). We set $L := \mathcal{O}_X(-A)$ and let $p : L \longrightarrow X$ be the bundle projection. Let B be the unit disk bundle associated with the hermitian line bundle (L, h_A^{-1}). Then B is a family of strongly pseudoconvex manifolds over Δ. We denote the bundle projection $B \longrightarrow X$ again by p and we shall identify X with the zero section of L. Let h_B be a C^∞-hermitian metric on $K_B + X$. Let dV_L a C^∞-volume form on L and let dV_B be the restriction of dV_L to B. We define (the diagonal part of) the m-th Bergman kernel $K_m(B)$ of $(K_B + X, h_B)$ by

$$K_m(B) = \sum_{i=1}^{\infty} \mid \phi_i^{(m)} \mid^2,$$

where $\{\phi_i^{(m)}\}_{i=1}^{\infty}$ is a complete orthonormal basis of the Hilbert space $\mathcal{H}_m$ of L^2-holomorphic sections of $m(K_B + L)$ with respect to h_B^m and dV_B and $\mid \phi_i^{(m)} \mid^2 := \phi_i^{(m)} \cdot \overline{\phi_i^{(m)}}$.

Then as in [15], we have the following lemma.

Lemma 4.1 *([15, p.256, Theorem 1.1], see also [16, Section 3.2])*

$$K_\infty(B) := \textit{the uppersemicontinuous envelope of } \overline{\lim}_{m\to\infty} \sqrt[m]{K_m(B)}$$

exists and

$$h_{B,\infty} = \frac{1}{K_{\infty,B}}.$$

is an AZD of $K_B + X$.

We note that $K_\infty(B)$ may be totally different from $\overline{\lim}_{m\to\infty} \sqrt[m]{K_m(B)}$ on X, since we have taken the uppersemicontinuous envelope. Let $\hat{X}$ denote the formal completion of B along X. Then we see that

$$H^0(\hat{X}, \mathcal{O}_{\hat{X}}(m(K_B + X))) = \oplus_{\ell \geq 0} H^0(X, \mathcal{O}_X(mK_X + \ell A))$$

holds. We note that $h_{B,\infty} \mid_X$ and $h_{B,\infty} \mid_{X_0}$ are well defined, since $\kappa(X_t) \geq 0$ for every $t \in \Delta$. In fact by the assumption, there exists a positive integer m_0 such that there exists a nonzero section $\sigma_0 \in H^0(X, \mathcal{O}_X(m_0 K_X))$ which does not vanish identically on every fiber. Then $p^*\sigma_0$ is identified with a global holomorphic section of $\mathcal{O}_B(m_0(K_B + X))$ by the adjunction formula and the line bundle structure of $p : L \longrightarrow X$. Shrinking Δ, if necessary, we may

assume that $p^*\sigma_0$ is a bounded holomorphic section of $(m_0(K_B + X), h_B)$ on B. Let $B_m(1)$ denote the unit ball of $\mathcal{H}_m$. We note that

$$K_m(B)(z) = \sup_{\sigma \in B_m(1)} \mid \sigma \mid^2 (z) \qquad (z \in B)$$

holds by definition. Then we see that there exists a positive constant c_0 such that

$$K_\infty(B) \geq c_0 \cdot \mid p^*\sigma_0 \mid^{2/m_0}$$

holds on B. Hence $h_{B,\infty} \mid_X$ and $h_{B,\infty} \mid_{X_0}$ are well defined.

This implies that $h \mid_{X_0}$ is well defined by the definition of a canonical AZD.

Lemma 4.2

$$\bar{\mathcal{I}}(h^m) = \bar{\mathcal{I}}(h^m_{B,\infty} \mid_X)$$

and

$$\bar{\mathcal{I}}(h^m_0) = \bar{\mathcal{I}}(h^m) \mid_{X_0} = \bar{\mathcal{I}}(h^m_{B,\infty} \mid_{X_0})$$

hold for every $m \geq 0$.

Proof of Lemma 4.2. We see that

$$\bar{\mathcal{I}}(h^m_{B,\infty} \mid_X) \subseteq \bar{\mathcal{I}}(h^m) \mid_X$$

holds for every $m \geq 0$ by the construction of h.

On the other hand we see that

$$(\star) \qquad \bar{\mathcal{I}}(h^m_{B,\infty} \mid_X) = \bar{\mathcal{I}}_X(h^m_{B,\infty})$$

holds by Theorem 2.3, where

$$\mathcal{I}_X(h^m_{B,\infty}) := \cap_{\ell \geq 1} \sqrt[\ell]{\mathcal{I}(h^{m\ell}_{B,\infty}) \mid_D}.$$

Let h_m be a canonical AZD of $mK_X + A$. Since A is ample, we see that

$$\bar{\mathcal{I}}(h^{m\ell}) \subseteq \bar{\mathcal{I}}(h_{m\ell})$$

holds for every $m, \ell \geq 0$. We note that for every $m, \ell \geq 0$,

$$\mathcal{O}_X(m\ell K_X + 2A) \otimes \mathcal{I}(h_{m\ell})$$

is generated by global sections by the definition of A. Hence shrinking Δ if necessary, we may and do assume that for every $m, \ell \geq 0$,

$$\mathcal{O}_X(m\ell K_X + 2A) \otimes \mathcal{I}(h_{m\ell})$$

is generated by

$$H^0_{(2)}(X, \mathcal{O}_X(m\ell K_X + 2A) \otimes \mathcal{I}(h_{m\ell})).$$

Hence noting the identity

$$H^0(\hat{X}, \mathcal{O}_{\hat{X}}(m(K_B + X))) = \oplus_{k\geq 0} H^0(X, \mathcal{O}_X(mK_X + kA)) \qquad (m \geq 0),$$

by the definition of $h_{B,\infty}$, we see that

$$\bar{\mathcal{I}}(h^m_{B,\infty}) \mid_D \supseteq \overline{\cap_{\ell\geq 1} \sqrt[\ell]{\mathcal{I}(h_{m\ell})}}$$

holds. By the identity ($\star$), this implies that the opposite inclusion

$$\bar{\mathcal{I}}(h^m) \subseteq \bar{\mathcal{I}}(h^m_{B,\infty} \mid_X)$$

holds for every $m \geq 0$. Hence we conclude that the equality :

$$\bar{\mathcal{I}}(h^m) = \bar{\mathcal{I}}(h^m_{B,\infty} \mid_X)$$

holds for every $m \geq 0$.

On the other hand, by Lemma 3.1 and ($\star$), we see that for every m

$$\bar{\mathcal{I}}(h^m_{B,\infty} \mid_{X_0}) \supseteq \overline{\cap_{\ell\geq 1} \sqrt[\ell]{\mathcal{I}(h^{m\ell}_{\ell,0})}}$$

holds, where $h_{m,0}$ denotes a canonical AZD of $mK_{X_0} + A$ on X_0. We note that obviously

$$\bar{\mathcal{I}}(h^{m\ell}_0) \subseteq \bar{\mathcal{I}}(h^\ell_{m,0})$$

holds for every $m \geq 1$ and $\ell \geq 0$. Hence we see that

$$\bar{\mathcal{I}}(h^m_0) \subseteq \bar{\mathcal{I}}(h^m_{B,\infty} \mid_{X_0})$$

holds. On the other hand, since

$$\bar{\mathcal{I}}(h^\ell_{B,\infty}) = \bar{\mathcal{I}}(h^\ell)$$

holds for every $\ell \geq 0$ as we have seen above, by Theorem 2.3 we obtain that

$$\bar{\mathcal{I}}(h^m_{B,\infty} \mid_{X_0}) = \bar{\mathcal{I}}(h^m \mid_{X_0})$$

holds for every $m \geq 0$.

Combining the above formulas, we see that

$$\bar{\mathcal{I}}(h^m_0) \subseteq \bar{\mathcal{I}}(h^m_{B,\infty} \mid_{X_0}) = \bar{\mathcal{I}}(h^m \mid_{X_0})$$

hold. We note that by definition

$$\bar{\mathcal{I}}(h^m \mid_{X_0}) \subseteq \bar{\mathcal{I}}(h^m_0)$$

holds for every $m \geq 0$. Hence

$$\bar{\mathcal{I}}(h^m_0) = \bar{\mathcal{I}}(h^m \mid_{X_0}) = \bar{\mathcal{I}}(h^m_{B,\infty} \mid_{X_0})$$

hold for every $m \geq 0$. This completes the proof of Lemma 4.2. **Q.E.D.**

4.2 Completion of the proof of Theorem 1.1

By the L^2-extension theorem ([11, p.200, Theorem]), we see that

$$H^0(X, \mathcal{O}_X(mK_X) \otimes \mathcal{I}(h^{m-1}_{B,\infty} \mid_X)) \to H^0(X_0, \mathcal{O}_{X_0}(mK_{X_0}) \otimes \mathcal{I}(h^m_{B,\infty} \mid_{X_0}))$$

is surjective for every $m \geq 1$.

On the other hand, we note that

$$H^0(X_0, \mathcal{O}_{X_0}(mK_{X_0})) \simeq H^0(X_0, \mathcal{O}_{X_0}(mK_{X_0}) \otimes \mathcal{I}_\infty(h^m_0))$$

holds for every $m \geq 0$ by the construction of the canonical AZD h_0 (cf. Remark 2.1). Since

$$\mathcal{I}_\infty(h^m_0) \subseteq \bar{\mathcal{I}}(h^m_0) \subseteq \mathcal{I}(h^m_0)$$

hold for every $m \geq 0$ and the closure of multiplier ideal sheaves are determined by the Lelong number of the pullback of the curvature current on some modifications as in Section 2.1, by Lemma 4.2

$$H^0(X_0, \mathcal{O}_{X_0}(mK_{X_0})) \simeq H^0(X_0, \mathcal{O}_{X_0}(mK_{X_0}) \otimes \mathcal{I}(h^m_{B,\infty} \mid_{X_0}))$$

holds for every $m \geq 0$. Hence we have that for every $m \geq 1$

$$P_m(X_0) \leq P_m(X_t)$$

holds for every $t \in \Delta$. This means that $P_m(X_t)$ is lowersemicontinuous as a function on $t \in \Delta$. By the uppersemicontinuity theorem of cohomology groups, we see that $P_m(X_t)$ is constant on Δ. This completes the proof of Theorem 1.1.

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